



**You have downloaded a document from
RE-BUS
repository of the University of Silesia in Katowice**

Title: Star-shaped set inversion fractals

Author: Krzysztof Gdawiec

Citation style: Gdawiec Krzysztof. (2014). Star-shaped set inversion fractals. "Fractals" (2014, iss. 4, [7 s.]), doi 10.1142/S0218348X14500091

Version: pre-print



Uznanie autorstwa - Użycie niekomercyjne - Bez utworów zależnych Polska - Licencja ta zezwala na rozpowszechnianie, przedstawianie i wykonywanie utworu jedynie w celach niekomercyjnych oraz pod warunkiem zachowania go w oryginalnej postaci (nie tworzenia utworów zależnych).



UNIwersYTET ŚLĄSKI
W KATOWICACH



Biblioteka
Uniwersytetu Śląskiego



Ministerstwo Nauki
i Szkolnictwa Wyższego

Star-shaped Set Inversion Fractals *

Krzysztof Gdawiec

Institute of Computer Science
University of Silesia
Będzińska 39, 41-200 Sosnowiec, Poland
kgdawiec@ux2.math.us.edu.pl

Abstract

In the paper, we generalized the idea of circle inversion to star-shaped sets and used the generalized inversion to replace the circle inversion transformation in the algorithm for the generation of the circle inversion fractals. In this way, we obtained the star-shaped set inversion fractals. The examples that we have presented show that we were able to obtain very diverse fractal patterns by using the proposed extension and that these patterns are different from those obtained with the circle inversion method. Moreover, because circles are star-shaped sets, the proposed generalization allows us to deform the circle inversion fractals in a very easy and intuitive way.

Keywords: fractal, circle inversion, star-shaped polygon

1 INTRODUCTION

The concept of fractals was first presented by Mandelbrot in the 1970s.¹ Based on the hypothesis presented by Mandelbrot, Barnsley presented some revolutionary ideas that emphasized the practical aspect of fractals. He provided methods to model natural fractals and used the concept of the iterated function system (IFS) as a tool to generate them.² Since then, fractals have been used in many applications, e.g., pattern recognition,³ image processing,⁴ computer aided geometric design,⁵ computer graphics,⁶ image synthesis,⁷ and even in medicine⁸ and archaeology.⁹

In the context of computer graphics and in other areas, fractals have been used to generate very complex and beautiful patterns.^{10–12} While fractal patterns are very complex, only a small amount of information is

*Electronic version of an article published as Fractals 22(4), 1450009, (2014), DOI: 10.1142/S0218348X14500091 ©World Scientific Publishing Company
<http://www.worldscientific.com/worldscinet/fractals>

needed to generate them, e.g., in the IFS, only information about a finite number of contractive mappings is needed. In the literature, there are many methods of generating fractal patterns, including deterministic algorithm and random iteration algorithm,² the escape time algorithm,¹³ and even the IFS ray tracing algorithm.¹⁴ In 2000, Frame and Cogevena introduced a method called circle inversion fractals¹⁵ that was based on circle inversion transformation, a well-known concept in geometry.¹⁶ In this paper, we showed how to extend the idea of circle inversion to any star-shaped set, and we provided some examples of fractals obtained with the star-shaped set inversion.

The paper is organized as follows: Sec. 2 is devoted to the definition of circle inversion and some background about circle inversion fractals. In Sec. 3, first we generalize the circle inversion transformation to star-shaped sets. Then, we present an algorithm used for the generation of the star-shaped set inversion fractals and give a method that allows the deformation of the original circle inversion fractals. In Sec. 4 we present some examples of deformed circle inversion fractals and star-shaped set inversion fractals. Our conclusions are presented in Sec. 5.

2 CIRCLE INVERSION

The inversion with respect to a circle (circle inversion) was introduced in Apollonius of Prega's book entitled *Plane Loci*.¹⁷ Since Apollonius introduced circle inversion, many applications of the concept have been used in geometry.¹⁶

Definition 1. Let C be a circle with a centre o and a radius R , and let p be any point other than o . If p' is the point on the ray $r(t) = o + t(p - o)$, where $t \in [0, \infty)$, that satisfies the equation:

$$d(o, p) \cdot d(o, p') = R^2, \quad (1)$$

where $d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$ is an Euclidean metric, then we call p' the *inverse* of p with respect to the circle C (Fig. 1). The point o is called the *centre of inversion*, and C is called the *circle of inversion*. The transformation that takes p and transforms it into p' is called the *circle inversion transformation*, and we denote it by I_C .

From relationship (1), one can derive an algebraic form of the circle inversion transformation. If we let $o = (x_o, y_o)$ and $p = (x_p, y_p)$, then I_C has following formula:¹⁵

$$p' = I_C(p) = (x_o, y_o) + \frac{R^2}{(x_p - x_o)^2 + (y_p - y_o)^2} (x_p - x_o, y_p - y_o). \quad (2)$$

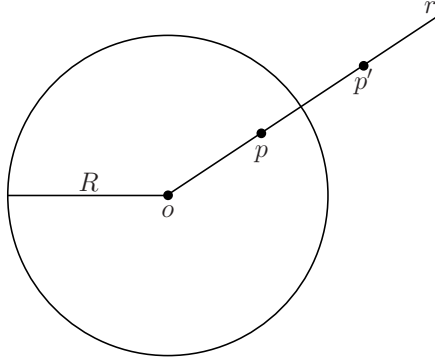


Figure 1: Inversion of p with respect to the circle that has centre o and radius R .

In definition 1, we assume that p is a point other than o , but we can extend the definition also to o .¹⁵ If $p = o$, then $I_C(o) = \infty$, and if $p = \infty$, then $I_C(\infty) = o$. In this way, I_C is defined on $\hat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$.

In 2000, Frame and Cogevina¹⁵ used the circle inversion transformation and introduced circle inversion fractals, and they proposed two algorithms for generation of those fractals. The algorithms were analogous to the deterministic and random algorithms for the iterated function systems introduced by Barnsley.² A modified version of the random algorithm is presented in Sec. 3. Leys expanded the idea of circle inversion fractals to spheres¹⁸ and obtained interesting 3D fractal patterns.

3 STAR-SHAPED SET INVERSION FRACTALS

In this section, we show that we can extend the idea of circle inversion to any star-shaped set. Let us start with some definitions about polygons from computational geometry.^{19,20}

Definition 2 (²¹). A simple polygon P is *star-shaped* if there exists a point z not external to P such that for all points p in P the line segment \overline{zp} lies entirely within P . The locus of the points z having the above property is the *kernel* of P .

Notice that every convex polygon is a star-shaped polygon and its interior is its kernel.²⁰ The kernel of a star-shaped polygon is always convex.²⁰ Figure 2 shows an example of a star-shaped polygon with its kernel.

We can extend the concept of the star-shaped polygon to a set as follows:

Definition 3. A set S in \mathbb{R}^2 is *star-shaped* if there exists a point $z \in \text{int } S$ ($\text{int } S$ means the interior of S) such that for all points $p \in S$ the line segment \overline{zp} lies entirely within S . The locus of the points z having the above property is the *kernel* of S .

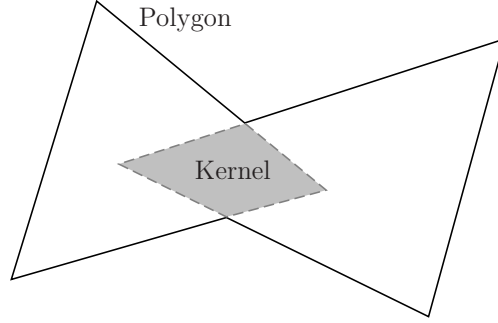


Figure 2: Star-shaped polygon and its kernel.

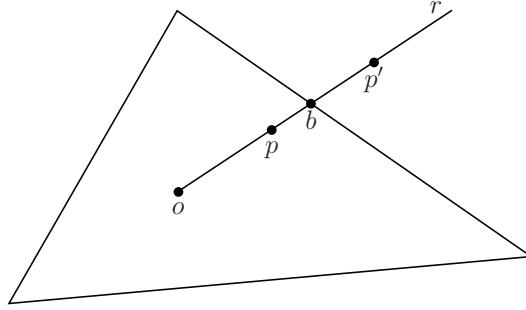


Figure 3: Inversion of p with respect to a given star-shaped set (triangle).

Now, let us assume that we have a star-shaped set S and some point o that belongs to the kernel of S . Moreover, let us assume that we have a point p other than o . We want to define the inverse of p with respect to S . For this purpose, we shoot a ray r from o in direction $p - o$, i.e., $r(t) = o + t(p - o)$ for $t \in [0, \infty)$, and find its intersection b with the boundary of S (Fig. 3). Because S is star-shaped and o belongs to its kernel, point b is unique. Point p' is said to be the *inverse* of p with respect to S if it satisfies the following equation:

$$d(o, p) \cdot d(o, p') = [d(o, b)]^2. \quad (3)$$

Similar to the case of the circle inversion, point o is called the *centre of inversion*. The transformation that takes p and transforms it into p' is called the *star-shaped set inversion transformation*, and it is denoted by I_S . Of course, in a similar way, we can extend the definition of I_S to o , so $I_S(o) = \infty$ and $I_S(\infty) = o$. As can be seen from the above construction, the star-shaped set inversion transformation is given by both the set S and the centre of inversion o , which can be different for the same set S .

The algebraic formula for I_S has a very similar form to the formula of

I_C . If we let $o = (x_o, y_o)$ and $p = (x_p, y_p)$, then I_S has the following form:

$$p' = I_S(p) = (x_o, y_o) + \frac{[d(o, b)]^2}{(x_p - x_o)^2 + (y_p - y_o)^2} (x_p - x_o, y_p - y_o). \quad (4)$$

Notice that S divides $\hat{\mathbb{R}}^2$ into three components, i.e., $B = \text{int } S$ (bounded component), $U = \hat{\mathbb{R}}^2 \setminus S$ (unbounded component), and ∂S . The star-shaped set inversion transformation has properties that are similar to those of the circle inversion transformation:

1. I_S interchanges B and U ,
2. I_S is an identity on C ,
3. I_S is a contraction on U and an expansion on B ,
4. I_S is an involution, i.e., $I_S(I_S(p)) = p$ for all $p \in \hat{\mathbb{R}}^2$.

Now that we have the definition of the star-shaped set inversion transformation and its properties, we are ready to provide an algorithm for the generation of an approximation of the star-shaped set inversion fractals (Algorithm 1). The algorithm is a modified version of the random inversion algorithm proposed by Frame and Cogevina.¹⁵ The modification consists of using a star-shaped set inversion transformation instead of a circle inversion transformation.

Algorithm 1: Random inversion algorithm.

Input: S_1, \dots, S_k – star-shaped sets with chosen centres of inversion,
 p_0 – starting point external to S_1, \dots, S_k , $n > 20$ – number of iterations.

Output: Approximation of a restricted limit set (star-shaped sets inversion fractal).

```

1  $j = \text{random number from } \{1, \dots, k\}$ 
2  $p = I_{S_j}(p_0)$ 
3 for  $i = 2$  to  $n$  do
4    $l = \text{random number from } \{1, \dots, k\}$ 
5   while  $j = l$  or  $\text{inSet}(S_l, p)$  do
6      $l = \text{random number from } \{1, \dots, k\}$ 
7    $j = l$ 
8    $p = I_{S_j}(p)$ 
9   if  $i > 20$  then
10     $\text{Plot } p$ 

```

To generate the approximation of the fractal, a set of star-shaped set inversion transformations, a starting point that lies outside of all the star-shaped sets and the number of iterations is required. First, we randomly choose a transformation and use it to transform the starting point. Because the starting point lies outside the star-shaped set that defines the transformation, i.e., it belongs to the unbounded component U of the transformation, and because property 1, the transformed point will lie in the bounded component B of the transformation. Then, in each iteration, we randomly choose a transformation, but this time with two restrictions. The first restriction is that we do not want to transform the point with the transformation that was used in the previous iteration. This restriction follows from the fact that I_S is an involution (property 4). Without this restriction, the number of points that approximate the fractal could decrease, and we would have to make more iterations to obtain a good approximation. The second restriction is that we do not want to transform the point with a transformation for which the point lies inside the bounded component B . This restriction is a consequence of property 3. We want the transformation to be a contraction, not an expansion, because, if we allow the transformation to be an expansion, the transformed point could escape, and the approximation would become messier. Moreover, the contraction guarantees the convergence of the algorithm. When we have chosen the transformation, we transform with it the point from the previous iteration. As in the random iteration algorithm for the IFS, the first couple of points do not belong to the approximation, so they are skipped. In the algorithm, we skip the first 20 points.

In the algorithm, when we go to the limit with the iterations, the generated points (orbit of the starting point) will converge to a *restricted limit set*, i.e., a limit set of the orbit of a point, with the restriction that if some orbit point p_i lies in the star-shaped set S_j , then the next orbit point p_{i+1} cannot be $I_{S_j}(p_i)$. The term restricted limit set (for circles) was introduced by Clancy and Frame.²² The proof of the algorithm's convergence is similar to the proof for the algorithm for circle inversion given by Smith.²³

Notice that the fractal approximation obtained with the algorithm will lie entirely inside the region bounded by the star-shaped sets that define the transformations used in the algorithm. This follows from the second restriction and property 1. The second restriction gives us the guarantee that the point that will be transformed lies in the unbounded component of the transformation, and property 1 gives us the guarantee that, after transformation, the point will lie inside the bounded component. In this way, each generated point will lie inside a bounded component of some transformation used in the algorithm.

Naturally, circles are star-shaped sets. When the centre of the circle, which of course belongs to its kernel, is taken as the centre of inversion, the star-shaped set inversion transformation reduces to the circle inversion

transformation from Sec. 2. But we can take any point from the kernel, which, for a circle, is equal to its interior. The examples in the next section show that changing the centre of inversion of the circles gives the possibility of deforming the original circle inversion fractal.

In implementation of the star-shaped set transformation for simple shapes, such as polygons and circles we can use the well known algorithms of finding their intersection with a ray and the determination of whether a point lies in a given figure.²⁴ Also, when we model the boundary of the star-shaped set with polynomial curves, appropriate algorithms exist in the literature for finding the intersection with a ray and an inclusion test.²⁵ In the general case, this task can be difficult to solve and, consequently, difficult to implement.

In the paper, we only show the 2D transformation, but we can extend the above considerations to 3D. The sets used in 3D must have a similar properties to the property of the star-shaped sets in 2D.

4 EXAMPLES

In this section, we present some examples of the star-shaped set inversion fractals. On the left side of all of the figures we have the set of star-shaped sets and their centres of inversion, and the approximation of the fractal is on the right side of the figures. Each of the sets is plotted using a different colour, and the points of the fractal are coloured with the colour of the set that was used to obtain the point.

The first example, presented in Fig. 4, shows the effect of changing the centres of inversion in the circle inversion fractals. The fractal is given by eight circles. Figure 4(a) shows the original fractal obtained with the method of Frame and Cogevina. Figure 4(b) shows that the centres of inversion of the large circles were moved in the diagonal direction near the boundary of the circles. It can be seen that the shape of the fractal was altered and that it follows the points. Figure 4(c) shows that we changed the centres of inversion of the small circles, moving them near the boundary of the circles. Also, in this case, the shape follows the points. Figure 4(d) shows that we changed the centres of inversion of all the large circles and the centres of inversion of the two small circles (upper and lower positions in the figure, respectively). The centres of the large circles were moved asymmetrically. This resulted in the shape's losing its central symmetry, but it maintained its axial symmetry. If we had moved the centres of inversions totally in an asymmetrical way, then we would obtain an asymmetrical pattern. From these examples, it is apparent that the circle inversion fractals easily can be deformed.

In the second example, we change consecutive circles into squares to show that the shape of the sets defining the inversion transformations has

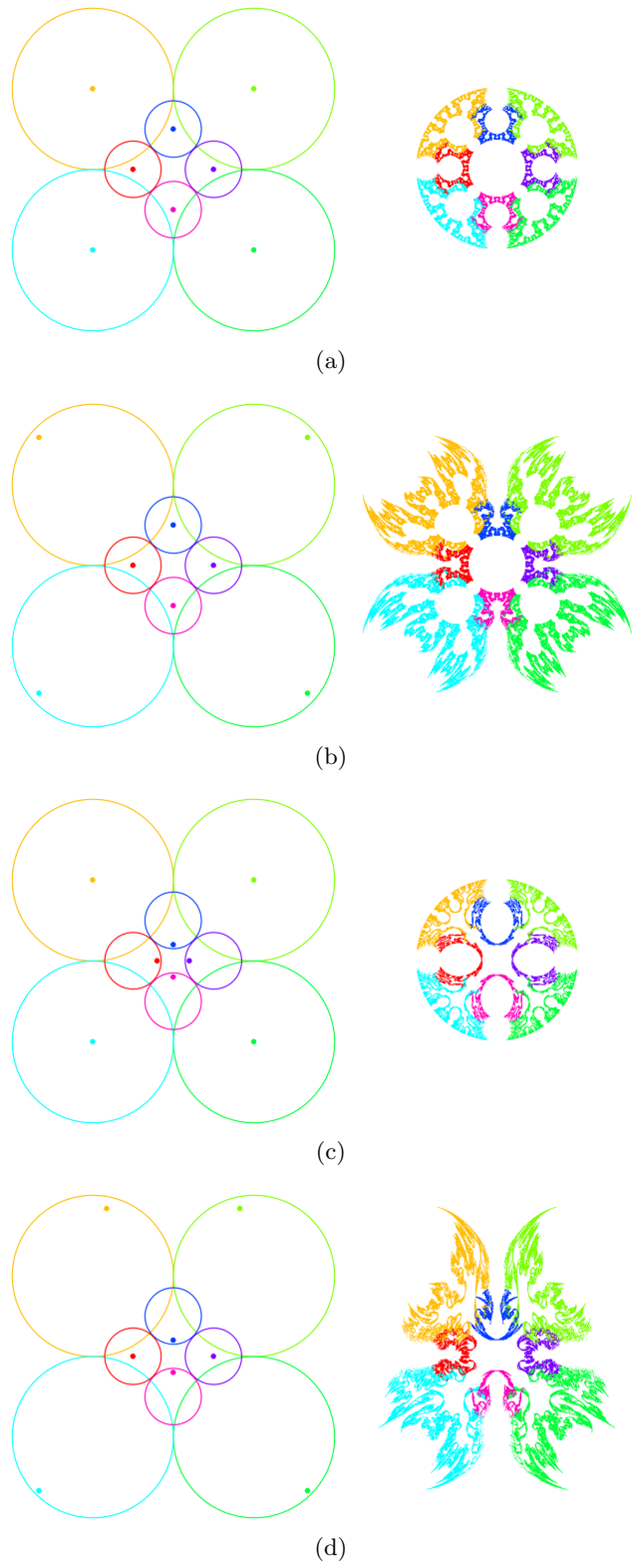


Figure 4: Examples of changing the centres of inversion in circles.

an effect on the shape of the obtained fractal, and Fig. 5 shows the results. In Fig. 5(a), the fractal is given by nine circles. The centres of inversion of the large circles were moved. In Fig. 5(b), the fractal is given by five circles and four squares. The four large circles from Fig. 5(a) were changed to squares inscribed in the large circles and the coordinates of the centres of inversion remained the same. Thus, the shape of the fractal was altered significantly. Next, we changed four additional circles into inscribed squares (Fig. 5(c)). This time, a smaller alternation of the shape was observed. The most significant change occurred in the centre of the fractal. The last circle was changed into a square in Fig. 5(d). And also, in this case, we see a small alteration of the shape that was easier to discern in the centre of the fractal.

Figure 6 shows various examples of star-shaped set inversion fractals. The fractal in Fig. 6(a) is defined by four triangles, four squares, and one circle. Figure 6(b) shows a fractal defined by four triangles, four star-shaped concave pentagons, and five circles. Figure 6(c) shows a fractal that is defined by more complex shapes than the previous two fractals. We used a cross that is defined by a polygon with 32 vertices and two stars defined by polygons with 24 vertices. The other shapes have simple geometries and consists of four right triangles and three circles. Figure 6(d) shows a fractal that is defined by four circles and ten sets with curved boundaries. From the examples, we see that we are able to obtain very diverse fractal patterns by using the star-shaped set transformation.

5 CONCLUSIONS

In this paper, we presented the concept of the generalization of the circle inversion transformation to star-shaped sets and, consequently, extending the algorithm for generating circle inversion fractals to star-shaped set inversion fractals. The use of the star-shaped sets gives us more possibilities for obtaining new and very interesting fractal patterns. Moreover, using the proposed generalization, we are able to deform the circle inversion fractals in an easy and intuitive way. The proposed generalization of the inversion transformation can be further generalized to \mathbb{R}^3 and used to extend the results of Leys.¹⁸

ACKNOWLEDGMENTS

The author thanks the anonymous referees for their careful reading, helpful comments, and many suggestions that improved the manuscript.

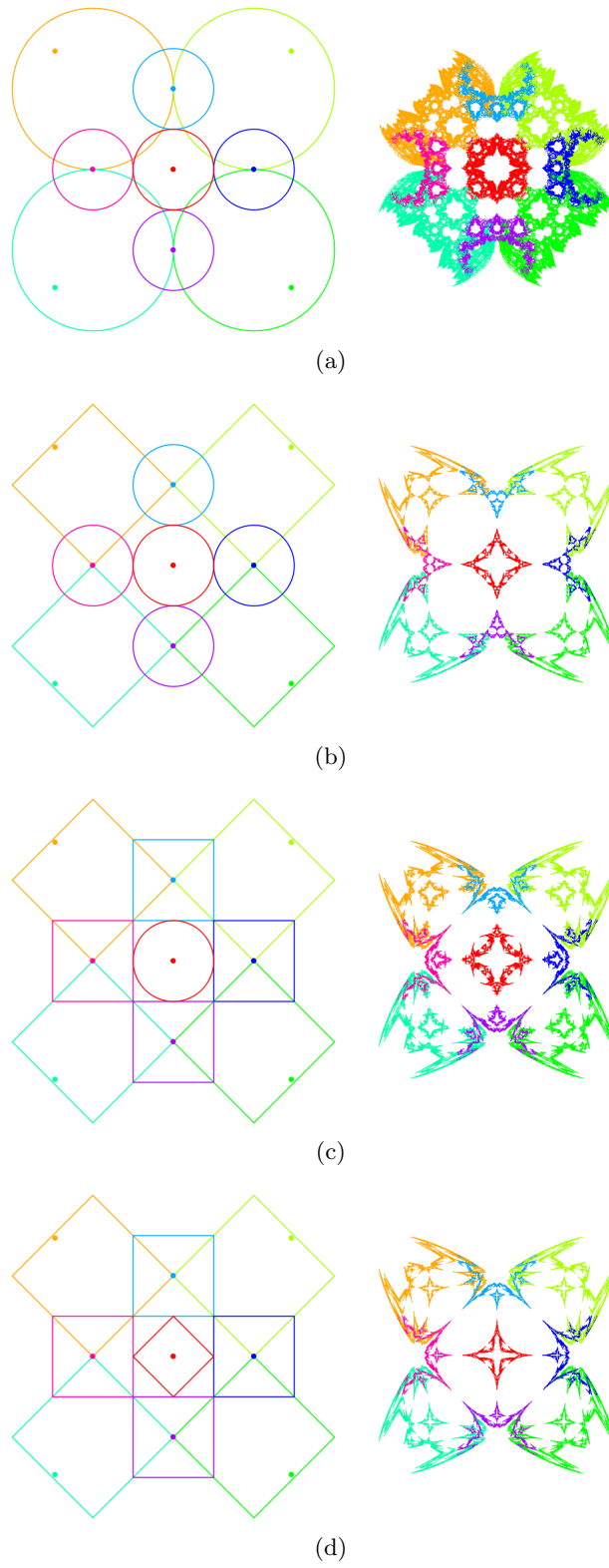
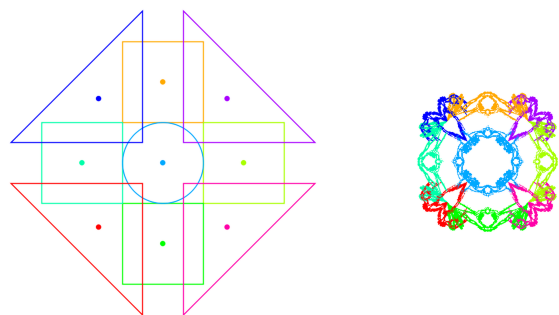
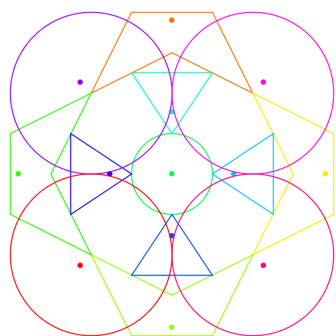


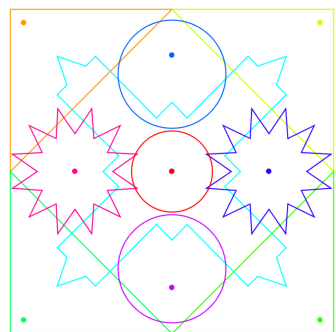
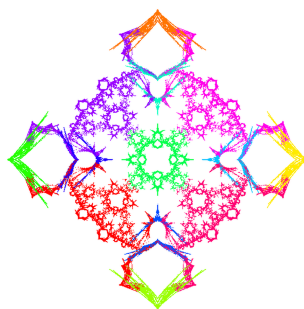
Figure 5: Examples of changing circles into squares in the star-shaped set inversion fractals.



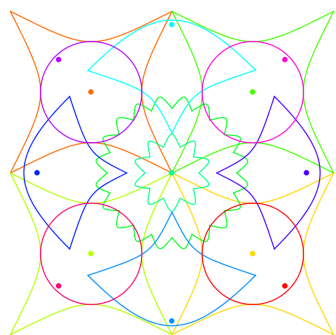
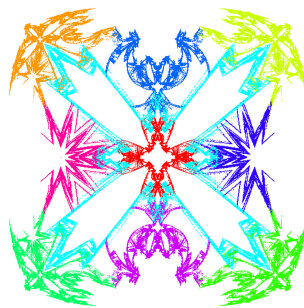
(a)



(b)



(c)



(d)

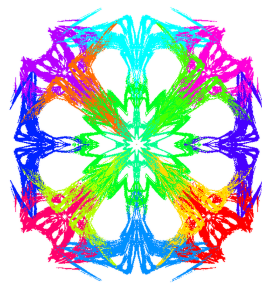


Figure 6: Examples of various star-shaped set inversion fractals.

REFERENCES

- [1] B. Mandelbrot, *The Fractal Geometry of Nature* (W.H. Freeman and Company, New York, 1983).
- [2] M. Barnsley, *Fractals Everywhere* (Academic Press, Boston, 1988).
- [3] K. Gdawiec and D. Domańska, Partitioned Iterated Function Systems with Division and a Fractal Dependence Graph in Recognition of 2D Shapes, *Int. J. Appl. Math. Comput. Sci.* **21**(4) (2011) 757–767.
- [4] M. Demers and H. Kunze, Fractal-based Methods in Imaging, *Nonlinear Anal.* **71**(12) (2009) e1413–e1419.
- [5] C. Gentil and M. Neveu, Mixed-aspect Fractal Surfaces, *Comput. Aided Des.* **45**(2) (2013) 432–439.
- [6] O. Belmonte, S. Sancho and J. Ribelles, Multiresolution Modeling using Fractal Image Compression Techniques, in *Theory and Practice of Computer Graphics*, edited by J. Collomosse, I. Grimstead (Eurographics Association, Sheffield, 2010) pp. 51–58.
- [7] S. Nikiel, *Iterated Function Systems for Real-time Image Synthesis* (Springer, London, 2007).
- [8] D. Lv, X. Guo, X. Wang, J. Zhang and J. Fang, Computerized Characterization of Prostate Cancer by Fractal Analysis in MR Images, *J. Magn. Reson. Imaging* **30**(1) (2009) 161–168.
- [9] C.T. Brown, W.R.T. Witschey. and L.S. Liebovitch, The Broken Past: Fractals in Archaeology, *J. Archaeol. Method Theory* **12**(1) (2005) 37–78.
- [10] J.C. Hart, L.H. Kauffman and D. J. Sandin, Interactive Visualization of Quaternion Julia Sets, in *Proceedings of the 1st Conference on Visualization '90* (IEEE Computer Society Press, San Francisco, California, 1990) pp. 209–218.
- [11] T. Martyn, Realistic Rendering 3D IFS Fractals in Real-time with Graphics Accelerators, *Comput. Graph.* **34**(2) (2010) 167–175.
- [12] J. Shier and P. Bourke, An Algorithm for Random Fractal Filling of Space, *Comput. Graph. Forum* **32**(8) (2013) 89–97.
- [13] D. Hepting and J.C. Hart, The Escape Buffer: Efficient Computation of Escape Time for Linear Fractals, in *Proceedings Graphics Interface '95* (Quebec, Canada, May 1995) pp. 204–214.

- [14] T. Martyn, An Approach to Ray Tracing Affine IFS Fractals, in *Emergent Nature*, edited by M.M. Novak (World Scientific, Singapore, 2002), pp. 283–292.
- [15] M. Frame and T. Cogevina, An Infinite Circle Inversion Limit Set Fractal, *Comput. Graph.* **24**(5) (2000) 797–804.
- [16] D.A. Brannan, M.F. Esplen and J.J. Gray, *Geometry*, 2nd ed. (Cambridge University Press, New York, 2012).
- [17] M. Frame, B. Mandelbrot and N. Neger, *Fractal Geometry* (2014) <http://classes.yale.edu/fractals/>
- [18] J. Leys, Sphere Inversion Fractals, *Comput. Graph.* **29**(3) (2005) 463–466.
- [19] J. O’Rourke, *Computational Geometry in C*, 2nd ed. (Cambridge University Press, New York, 1998).
- [20] S.K. Ghosh, *Visibility Algorithms in the Plane* (Cambridge University Press, New York, 2007).
- [21] F.P. Preparata and M.I. Shamos, *Computational Geometry: An Introduction* (Springer-Verlag, New York, 1985).
- [22] C. Clancy and M. Frame, Fractal Geometry of Restricted Sets of Circle Inversions, *Fractals* **3**(4) (1995) 689–699.
- [23] R. Smith, *Fractal Producing Iterative Mapping Systems on Circles*, M.S. thesis, Univeristy of Newcastle, Australia, 2010.
- [24] P.J. Schneider and D.H. Eberly, *Geometric Tools for Computer Graphics* (Morgan Kaufmann, San Francisco, 2003).
- [25] M.K. Agoston, *Computer Graphics and Geometric Modeling: Implementation and Algorithms* (Springer-Verlag, London, 2005).